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STABILITY AND BIFURCATION ANALYSIS IN A CLASS OF TWO-NEURON NETWORKS WITH DISTRIBUTED DELAYS

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Abstract

In this paper, a class of two-neuron networks with distributed delays is considered. By analyzing the associated characteristic transcendental equation, its linear stability is investigated and Hopf bifurcation is demonstrated. Some explicit formulae for determining the stability and the direction of the Hopf bifurcation periodic solutions bifurcating from Hopf bifurcations are obtained by using the normal form theory and center manifold theory.

1. Introduction

Based on the assumption that the elements in the network can respond to and communicate with each other instantaneously without time delays, Hopfield [7, 8] proposed Hopfield neural networks (HNNs) model in 1980s. During the past several years, the dynamical phenomena

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of neural networks have been extensively studied because of the wider application in various information processing, optimization problems, etc. In particular, the appearance of a cycle bifurcating from an equilibrium of an ordinary or a delayed neural network with a single parameter, which is known as a Hopf bifurcation, has attracted much attention, we can see [2, 5, 11]. It should be pointed out that a great number of papers study the Hopf bifurcation of neural networks with discrete delays. We know that the neural network with distributed delay is more general than that with discrete delay, because the distributed delay becomes a discrete delay when the delay kernel is a delta function at a certain time. Thus, investigating the dynamical behaviors of neural networks with distributed delay has theoretical and practical significance.

In 2007, Tang [14] investigated the Hopf bifurcation of the following two-neuron networks with distributed delay:

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + a_1 \tanh[x_2(t)] - a_1 b_1 \int_{-\infty}^{0} F(-s) \tanh[x_2(t+s)] ds, \\ \dot{x}_2(t) = -x_2(t) + a_2 \tanh[x_1(t)], \end{cases} \tag{1.1}$$

where $\alpha_1, \alpha_2, a_1, b_1$ are real parameters, and obtain a sufficient condition for a Hopf bifurcation to occur for system (1.1) by using the standard normal form theory and the center manifold theorem. In this paper, we consider the following more general neural networks with distributed delay:

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + a_1 \tanh[x_2(t)] - a_1 b_1 \int_{-\infty}^{0} F(-s) \tanh[x_2(t+s)] ds, \\ \dot{x}_2(t) = -x_2(t) + a_2 \tanh[x_1(t)] - a_2 b_2 \int_{-\infty}^{0} F(-s) \tanh[x_1(t+s)] ds, \end{cases} \tag{1.2}$$

where $\alpha_1, \alpha_2, a_i, b_j$ $(i, j = 1, 2)$ are real parameters. Obviously, if the coefficient a_2 or b_2 in system (1.2) is equal to zero, then system (1.2) becomes system (1.1). It should be pointed out that the analysis method on the characteristic equation in Tang [14] can not be applied to counterpart in the system (1.2). So, our work of dealing with the Hopf bifurcation of system (1.2) has theoretical and practical meanings.

The purpose of this paper is to discuss the stability and the properties of Hopf bifurcation of model (1.2). To the best of our knowledge, it is the first to deal with the stability and Hopf bifurcation of system (1.2).

This paper is organized as follows. In Section 2, the existence of Hopf bifurcation at the equilibrium is studied. In Section 3, the direction of Hopf bifurcation and the stability and periodic of bifurcating periodic solutions on the center manifold are determined. Some main conclusions are drawn in Section 4.

2. Existence of Hopf Bifurcation

Throughout this paper, we assume that

(H1) $|a_1 a_2 (1 - b_1)(1 - b_2)| < 1$ and $uf(u) > 0$, for $u \neq 0$.

(H2) The function $F(s)$ is strong kernel, i.e., $F(s) = \alpha^2 s e^{-\alpha s}$, $(\alpha > 0)$.

The hypothesis (H1) implies that $E_*(0, 0)$ is a unique equilibrium of system (1.2) and linearized system of (1.2) takes the form:

$$\frac{dX}{dt} = LX(t) + \int_{-\infty}^{0} K(s)X(t+s)ds + H(x), \tag{2.1}$$

where $X = (x_1(t), x_2(t))^T$,

$$L = \begin{pmatrix} -1 & a_1 \\ a_2 & -1 \end{pmatrix}, \; K(s) = \begin{pmatrix} 0 & -a_1 b_1 F(-s) \\ -a_2 b_2 F(-s) & 0 \end{pmatrix}, \tag{2.2}$$

and

$$H(x) = \begin{pmatrix} -\frac{a_1}{3} x_2^3(t) + \frac{a_1 b_1}{3} \int_{-\infty}^{0} F(-s) x_2^3(t+s) ds + \cdots \\ -\frac{a_2}{3} x_1^3(t) + \frac{a_2 b_2}{3} \int_{-\infty}^{0} F(-s) x_1^3(t+s) ds + \cdots \end{pmatrix}.$$

The associated characteristic equation of (2.1) is

$$(\lambda+1)^2 - a_1a_2\left[(1-b_1)\int_{-\infty}^{0} F(-s)e^{\lambda s}ds\right]\left[(1-b_2)\int_{-\infty}^{0} F(-s)e^{\lambda s}ds\right] = 0. \tag{2.3}$$

By the assumption (H2), (2.3) becomes the following polynomial equation of six degree:

$$(\lambda+1)^2(\lambda+\alpha)^4 - a_1a_2[(\lambda+\alpha)^2 - b_1\alpha^2][(\lambda+\alpha)^2 - b_2\alpha^2] = 0,$$

that is,

$$\lambda^6 + l_1(\alpha)\lambda^5 + l_2(\alpha)\lambda^4 + l_3(\alpha)\lambda^3 + l_4(\alpha)\lambda^2 + l_5(\alpha)\lambda + l_6 = 0, \tag{2.4}$$

where

$$\begin{aligned}
l_1(\alpha) &= 4\alpha + 2,\\
l_2(\alpha) &= 4\alpha^2 + 8\alpha + 3 - a_1a_2,\\
l_3(\alpha) &= 8\alpha^2 + 8\alpha + 4,\\
l_4(\alpha) &= [a_1a_2(b_1+b_2)+4]\alpha^2 + (8-4a_1a_2)\alpha + 2a_1a_2 + 3,\\
l_5(\alpha) &= 4\alpha + 2,\\
l_6(\alpha) &= -a_1a_2[(2\alpha+1)^2 - \alpha^2(b_1+b_2)(2\alpha+1) + b_1b_2\alpha^4] + 1.
\end{aligned}$$

By the Routh-Hurwitz criterion, we can easily obtain the following result:

Lemma 2.1. *Under the assumptions* (H1) *and* (H2), *then the equilibrium* $E_*(0, 0)$ *of system* (1.2) *is locally asymptotically stable, if the following conditions hold*:

$$D_1(\alpha) = 4\alpha + 2 > 0, \tag{2.5}$$

$$D_2(\alpha) = \begin{vmatrix} l_1(\alpha) & 1 \\ l_3(\alpha) & l_2(\alpha) \end{vmatrix} > 0, \tag{2.6}$$

$$D_3(\alpha) = \begin{vmatrix} l_1(\alpha) & 1 & 0 \\ l_3(\alpha) & l_2(\alpha) & l_1(\alpha) \\ l_5(\alpha) & l_4(\alpha) & l_3(\alpha) \end{vmatrix} > 0, \tag{2.7}$$

$$D_4(\alpha) = \begin{vmatrix} l_1(\alpha) & 1 & 0 & 0 \\ l_3(\alpha) & l_2(\alpha) & l_1(\alpha) & 1 \\ l_5(\alpha) & l_4(\alpha) & l_3(\alpha) & l_2(\alpha) \\ l_7(\alpha) & l_6(\alpha) & l_5(\alpha) & l_4(\alpha) \end{vmatrix} > 0, \tag{2.8}$$

$$D_5(\alpha) = \begin{vmatrix} l_1(\alpha) & 1 & 0 & 0 & 0 \\ l_3(\alpha) & l_2(\alpha) & l_1(\alpha) & 1 & 0 \\ l_5(\alpha) & l_4(\alpha) & l_3(\alpha) & l_2(\alpha) & l_1(\alpha) \\ l_7(\alpha) & l_6(\alpha) & l_5(\alpha) & l_4(\alpha) & l_3(\alpha) \\ l_9(\alpha) & l_8(\alpha) & l_7(\alpha) & l_6(\alpha) & l_5(\alpha) \end{vmatrix} > 0, \tag{2.9}$$

$$D_6(\alpha) = l_6(\alpha)D_5(\alpha) > 0, \tag{2.10}$$

where $l_7(\alpha) = l_8(\alpha) = l_9(\alpha) = 0$.

Let $\lambda = i\omega_0$, $\alpha = \alpha_0$, which are substituted into the Equation (2.4). For the sake of simplicity, we denote ω_0 and α_0 by ω, α, respectively. Then (2.4) becomes

$$-\omega^6 + l_1(\alpha)\omega^5 i + l_2(\alpha)\omega^4 - l_3(\alpha)\omega^3 i - l_4(\alpha)\omega^2 + l_5(\alpha)\omega i + l_6(\alpha) = 0.$$

Separating the real and imaginary parts, we get

$$\begin{cases} \omega^6 - l_2(\alpha)\omega^4 + l_4(\alpha)\omega^2 - l_6(\alpha) = 0, \\ l_1(\alpha)\omega^5 - l_3(\alpha)\omega^3 + l_5(\alpha)\omega = 0. \end{cases} \tag{2.11}$$

Let $z = \omega^2 (\omega > 0)$, then Equation (2.11) can be rewritten as

$$\begin{cases} z^3 - l_2(\alpha)z^2 + l_4(\alpha)z - l_6(\alpha) = 0, \\ l_1(\alpha)z^2 - l_3(\alpha)z + l_5(\alpha) = 0. \end{cases} \tag{2.12}$$

In order to investigate the bifurcation of system (1.2), now we make the following assumption:

(H3) Suppose (2.12) has at least a set of positive real roots.

Taking the derivative of λ with respect to α in (2.4), we have

$$\frac{d\lambda}{d\alpha} = -\frac{l_1'(\alpha)\lambda^5 + l_2'(\alpha)\lambda^4 + l_3'(\alpha)\lambda^3 + l_4'(\alpha)\lambda^2 + l_5'(\alpha)\lambda + l_6'(\alpha)}{6\lambda^5 + 5l_1(\alpha)\lambda^4 + 4l_2(\alpha)\lambda^3 + 3l_3(\alpha)\lambda^2 + 2l_4(\alpha)\lambda + l_5(\alpha)},$$

where $l_i' = \dfrac{dl_i}{d\alpha}$, $(i = 1, 2, 3, 4, 5, 6)$, and

$$\begin{aligned}
l_1'(\alpha) &= 4,\\
l_2'(\alpha) &= 8\alpha + 8,\\
l_3'(\alpha) &= 16\alpha + 8,\\
l_4'(\alpha) &= 8\alpha - 4a_1a_2 + 2a_1a_2(b_1 + b_2)\alpha + 8,\\
l_5'(\alpha) &= 4,\\
l_6'(\alpha) &= -a_1a_2[4(2\alpha + 1) - (b_1 + b_2)(6\alpha^2 + 2\alpha) + 4b_1b_2\alpha^3].
\end{aligned}$$

Thus

$$\frac{d\lambda}{d\alpha} = -\frac{(ME + NF) + i(MF - NE)}{M^2 + N^2},$$

where

$$\begin{aligned}
M &= 5l_1(\alpha)\omega^4 - 3l_3(\alpha)\omega^2 + l_5(\alpha),\\
N &= 6\omega^5 - 4l_4(\alpha)\omega^3 + 2l_4(\alpha)\omega,\\
E &= l_2'(\alpha)\omega^4 - l_4'(\alpha)\omega^2 + l_6'(\alpha),\\
F &= l_1'(\alpha)\omega^5 - l_3'(\alpha)\omega^3 + l_5'(\alpha).
\end{aligned}$$

So, we obtain

$$\operatorname{Re}\left[\frac{d\lambda}{d\alpha}\right]_{\alpha=\alpha_0,\,\lambda=i\omega_0} = -\frac{M_0E_0 + N_0F_0}{M_0^2 + N_0^2},$$

where

$$M_0 = 5l_1(\alpha_0)\omega_0^4 - 3l_3(\alpha_0)\omega_0^2 + l_5(\alpha_0),$$

$$N_0 = 6\omega_0^5 - 4l_4(\alpha_0)\omega_0^3 + 2l_4(\alpha_0)\omega_0,$$

$$E_0 = l_2'(\alpha_0)\omega_0^4 - l_4'(\alpha_0)\omega_0^2 + l_6'(\alpha_0),$$

$$F_0 = l_1'(\alpha_0)\omega_0^5 - l_3'(\alpha_0)\omega_0^3 + l_5'(\alpha_0).$$

Now, we assume that

$$\text{(H4)} \qquad \operatorname{Re}\left[\frac{d\lambda}{d\alpha}\right]_{\alpha=\alpha_0,\,\lambda=i\omega_0} = -\frac{M_0E_0 + N_0F_0}{M_0^2 + N_0^2} \neq 0.$$

Based on the above analysis, we can easily obtain the following result.

Theorem 2.1. *For system* (1.2), *if the conditions* (2.5)-(2.10) *and* (H1)-(H4) *hold, then when the parameter* α *passes through* α_0, *a Hopf bifurcation occurs at the equilibrium* $E_*(0, 0)$.

3. Direction and Stability of the Hopf Bifurcation

In the previous section, we obtained some conditions, which guarantee that the two-neuron networks with resonant bilinear terms undergoes the Hopf bifurcation at some values of $\alpha = \alpha_0$. In this section, we shall derived the explicit formulae for determining the direction, stability, and period of these periodic solutions bifurcating from the equilibrium $E_*(0, 0)$ at these critical value of α, by using techniques from normal form and center manifold theory in Hassard et al. [6]. Throughout this section, we always assume that system (2.1) undergoes Hopf bifurcation at the equilibrium $E_*(0, 0)$ for $\alpha = \alpha_0$, and then $\pm i\omega_0$ is corresponding purely imaginary roots of the characteristic equation at the equilibrium $E_*(0, 0)$.

For convenience, let $\alpha = \alpha_0 + \mu$, $\mu \in R$. Then, $\mu = 0$ is the Hopf bifurcation value of (2.1). Thus, we shall study Hopf bifurcation of small amplitude periodic solutions of (2.1) from the equilibrium point for μ

close to 0. We can consider the fixed phase space $C = C([-\infty, 0], R^2)$. We first transform system (2.1) into an operator equation of the form:

$$\dot{x}_t = A(\mu)x_t + R(\mu)x_t, \tag{3.1}$$

where $x = (x_1, x_2)^T$, $x_t(\theta) = x(t+\theta)$, $\theta \in [-\infty, 0]$, and the operator A and R are defined as

$$A(\mu)\phi(\theta) = \begin{cases} \dfrac{d\phi(\theta)}{d\theta} & -\infty \le \theta < 0, \\ L\phi(\theta) + \displaystyle\int_{-\infty}^{0} K(s)\phi(s)ds, & \theta = 0, \end{cases} \tag{3.2}$$

and

$$R\phi = \begin{cases} (0, 0)^T, & -\infty \le \theta < 0, \\ (f_1(\mu, \phi), f_2(\mu, \phi))^T, & \theta = 0, \end{cases} \tag{3.3}$$

respectively, where L, K are defined by (2.2) and

$$\begin{aligned} f_1(\mu, \phi) &= (\alpha_1 + a)\left[\frac{f''(0)}{2}\phi_1^2(0) + \frac{f'''(0)}{3!}\phi_1^3(0)\right] \\ &\quad + (\alpha_2 + b)\left[\frac{f''(0)}{2}\phi_2^2(s) + \frac{f'''(0)}{3!}\phi_2^3(s)\right] + h.o.t., \\ f_2(\mu, \phi) &= (\alpha_2 - b)\left[\frac{f''(0)}{2}\phi_1^2(s) + \frac{f'''(0)}{3!}\phi_1^3(s)\right] \\ &\quad + (\alpha_1 - a)\left[\frac{f''(0)}{2}\phi_2^2(0) + \frac{f'''(0)}{3!}\phi_2^3(0)\right] + h.o.t.. \end{aligned}$$

For $\psi \in C([-\infty, 0], (R^2)^*)$, the operator A^* is defined as

$$A^*\psi(s) = \begin{cases} -\dfrac{d\psi(s)}{ds} & s \in (0, +\infty], \\ L^T\phi(s) + \displaystyle\int_{-\infty}^{0} K^T(s)\psi(-s)ds, & s = 0. \end{cases} \tag{3.4}$$

Since A and A^* can have complex eigenvectors, it is therefore reasonable to assume that $\phi, \psi : [0, +\infty) \to C^2$.

For $\phi \in C([-\infty, 0], R^2)$ and $\psi \in C([0, +\infty], (R^2)^*)$, define the bilinear form

$$< \psi, \phi > = \overline{\psi}(0)\phi(0) - \int_{-\infty}^{0}\int_{\xi=0}^{\theta} \overline{\psi}(\xi - \theta)K(\theta)\phi(\xi)d\xi d\theta, \tag{3.5}$$

where $\eta(\theta) = \eta(\theta, 0)$. Then, we easily obtain the following result on the relation between the operators $A = A(0)$ and A^*.

Lemma 3.1. *$A = A(0)$ and A^* are adjoint operators.*

The proof of Lemma 3.1 is easy by (3.5), so we omit it.

By the discussions in Section 2, we know that $\pm i\omega_0$ are eigenvalues of $A(0)$, and they are also eigenvalues of A^* corresponding to $i\omega_0$ and $-i\omega_0$, respectively. We have the following result.

Lemma 3.2. *The vector*

$$q(\theta) = (1, \gamma)^T e^{i\omega_0\theta}, \quad \theta \in (-\infty, 0],$$

where

$$\gamma = \frac{(i\omega_0 + 1)(i\omega_0 + \alpha)^2}{a_1[(i\omega_0 + \alpha)^2 - b_1\alpha^2]}$$

is the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0$, and

$$q^*(s) = D(1, \gamma^*)e^{i\omega s}, \quad s \in [0, +\infty),$$

where

$$\gamma^* = \frac{(1 - i\omega_0)(i\omega_0 + \alpha)^2}{a_2[(i\omega_0 + \alpha)^2 - b_2\alpha^2]}$$

is the eigenvector of A^* corresponding to the eigenvalue $-i\omega_0$, moreover, $< q^*(s), q(\theta) > = 1$, where

$$D = \frac{\alpha + i\omega_0}{(1 + a\bar{a}^*)(\alpha + i\omega_0) - 2\alpha^2(\bar{a}^* a_2 b_2 + \alpha a_1 b_1)}.$$

Proof. Let $q(\theta)$ be the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0$ and $q^*(s)$ be the eigenvector of A^* corresponding to the eigenvalue $-i\omega_0$, namely, $A(0)q(\theta) = i\omega_0 q(\theta)$ and $A^* q(s) = -i\omega_0 q^*(s)$. From the definitions of $A(0)$ and A^*, we have $A(0)q(\theta) = dq(\theta)/d\theta$ and $A^* q(s) = -dq^*(s)/ds$. Thus, $q(\theta) = q(0)e^{i\omega_0\theta}$ and $q^*(s) = q^*(0)e^{i\omega_0 s}$. In addition,

$$A(0)q(0) = Lq(0) + \int_{-\infty}^{0} K(s)q(s)ds = i\omega_0 q(0). \tag{3.6}$$

That is,

$$\begin{pmatrix} i\omega_0 + 1 & -a_1 + \int_{-\infty}^{0} a_1 b_1 F(-s)e^{-\lambda s}ds \\ -a_2 + \int_{-\infty}^{0} a_2 b_2 F(-s)e^{-\lambda s}ds & i\omega_0 + 1 \end{pmatrix} q(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.7}$$

Therefore, we can easily obtain

$$\gamma = \frac{i\omega_0 + 1}{a_1[1 - b_1\int_{-\infty}^{0} F(-s)e^{\lambda s}ds]} = \frac{(i\omega_0 + 1)(i\omega_0 + \alpha)^2}{a_1[(i\omega_0 + \alpha)^2 - b_1\alpha^2]},$$

and so

$$q(0) = \left(1, \frac{(i\omega_0 + 1)(i\omega_0 + \alpha)^2}{a_1[(i\omega_0 + \alpha)^2 - b_1\alpha^2]}\right)^T.$$

Hence

$$q(\theta) = \left(1, \frac{(i\omega_0 + 1)(i\omega_0 + \alpha)^2}{a_1[(i\omega_0 + \alpha)^2 - b_1\alpha^2]}\right)^T e^{i\omega_0\theta}.$$

On the other hand,

$$A^* q^*(0) = L^T q^*(0) + \int_{-\infty}^{0} K^T(s) q * (-s) ds = -i\omega_0 q^*(0). \tag{3.8}$$

Namely,

$$\begin{pmatrix} i\omega_0 - 1 & a_2 - \int_{-\infty}^{0} a_2 b_2 F(-s) e^{-\lambda s} ds \\ a_1 - \int_{-\infty}^{0} a_1 b_1 F(-s) e^{-\lambda s} ds & i\omega_0 - 1 \end{pmatrix} q^*(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.9}$$

Therefore, we can easily obtain

$$\gamma^* = \frac{1 - i\omega_0}{a_2[1 - b_2 \int_{-\infty}^{0} F(-s) e^{\lambda s} ds]} = \frac{(1 - i\omega_0)(i\omega_0 + \alpha)^2}{a_2[(i\omega_0 + \alpha)^2 - b_2 \alpha^2]},$$

and so

$$q^*(0) = \left(1, \frac{(1 - i\omega_0)(i\omega_0 + \alpha)^2}{a_2[(i\omega_0 + \alpha)^2 - b_2 \alpha^2]}\right).$$

Thus

$$q^*(s) = \left(1, \frac{(1 - i\omega_0)(i\omega_0 + \alpha)^2}{a_2[(i\omega_0 + \alpha)^2 - b_2 \alpha^2]}\right) e^{i\omega_0 s}.$$

In the sequel, we shall verify that $< q^*(s), q(\theta) > = 1$. In fact, from (3.9), we have

$$< q^*(s), q(\theta) >$$

$$= \overline{D}(1, \bar{\gamma}^*)(1, \gamma)^T - \int_{-\infty}^{0} \int_{\xi=0}^{\theta} \overline{D}(1, \bar{\gamma}^*) e^{-i\omega_0(\xi - \theta)} d\eta(\theta)(1, \gamma)^T e^{i\omega_0 \xi} d\xi$$

$$= \overline{D}\left[1 + \gamma\bar{\gamma}^* - \int_{-\tau_1^0}^{0} (1, \bar{\gamma}^*) \theta e^{i\omega_0 \theta} d\eta(\theta)(1, \gamma)^T \right]$$

$$= \overline{D}\left\{ 1 + \gamma\bar{\gamma}^* - (1, \bar{\gamma}^*)\,[B(-\tau_2^0)e^{-i\omega_0\tau_2^0} + C(-\tau_1^0)e^{-i\omega_0\tau_1^0}](1, \gamma)^T \right\}$$

$$= \overline{D}\left[1 + \gamma\bar{\gamma}^* + \bar{\gamma}^*\tau_2^0(\alpha_2 - b)f'(0)e^{i\omega_0\tau_2^0} + \gamma\tau_1^0(\alpha_2 + b)f'(0)e^{-i\omega_0\tau_1^0} \right]$$

$$= 1.$$

Next, we use the same notations as those in Hassard et al. [6], and we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let x_t be the solution of Equation (3.1) when $\mu = 0$.

Define

$$z(t) = < q^*, x_t >, \qquad W(t, \theta) = x_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}, \tag{3.10}$$

on the center manifold C_0, and we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta), \tag{3.11}$$

where

$$W(z(t), \bar{z}(t), \theta) = W(z, \bar{z}) = W_{20}\frac{z^2}{2} + W_{11}z\bar{z} + W_{02}\frac{\bar{z}^2}{2} + \cdots, \tag{3.12}$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$. Noting that W is also real if x_t is real, we consider only real solutions. For solutions $x_t \in C_0$ of (3.1),

$$\begin{aligned}
\dot{z}(t) &= < q^*(s), \dot{x}_t > = < q^*(s), A(0)x_t + R(0)x_t > \\
&= < q^*(s), A(0)x_t > + < q^*(s), R(0)x_t > \\
&= < A^*q^*(s), x_t > + \bar{q}^*(0)R(0)x_t \\
&\quad - \int_{-\infty}^{0}\int_{\xi=0}^{\theta} \bar{q}^*(\xi - \theta)d\eta(\theta)A(0)R(0)x_t(\xi)d\xi \\
&= < i\omega_0 q^*(s), x_t > + \bar{q}^*(0)f(0, x_t(\theta))
\end{aligned}$$

$$\stackrel{\text{def}}{=} i\omega_0 z(t) + \bar{q}^*(0) f_0(z(t), \bar{z}(t)). \tag{3.13}$$

That is,

$$\dot{z}(t) = i\omega_0 z + g(z, \bar{z}), \tag{3.14}$$

where

$$g(z, \bar{z}) = g_{20}\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots. \tag{3.15}$$

Hence, we have

$$g(z, \bar{z}) = \bar{q}^*(0) f_0(z, \bar{z}) = f(0, x_t) = \bar{D}(1, \gamma^*)(f_1(0, x_t), f_2(0, x_t))^T, \tag{3.16}$$

where

$$f_1(\mu, \phi) = -\frac{a_1}{3} x_{2t}^3(0) + \frac{a_1 b_1}{3}\int_{-\infty}^{0} F(-s) x_{2t}^3(0) ds + h.o.t.,$$

$$f_2(\mu, \phi) = -\frac{a_2}{3} x_{1t}^3(0) + \frac{a_2 b_2}{3}\int_{-\infty}^{0} F(-s) x_{1t}^3(0) ds + h.o.t..$$

Noticing $x_t(\theta) = (x_{1t}(\theta), x_{2t}(\theta))^T = W(t, \theta) + zq(\theta) + \bar{z}\,\bar{q}$ and $q(\theta) = (1, \gamma)^T e^{i\omega_0\theta}$, we have

$$x_{1t}(0) = z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots,$$

$$x_{2t}(0) = \gamma z + \bar{\gamma}\bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots,$$

$$x_{1t}(s) = e^{i\omega_0 s} z + e^{-i\omega_0}\bar{z} + W_{11}^{(1)}(s)\frac{z^2}{2} + W_{11}^{(1)}(s) z\bar{z} + W_{02}^{(1)}(s)\frac{\bar{z}^2}{2} + \cdots,$$

$$x_{2t}(s) = \gamma e^{i\omega_0 s} z + \bar{\gamma} e^{-i\omega_0}\bar{z} + W_{11}^{(2)}(s)\frac{z^2}{2} + W_{11}^{(2)}(s) z\bar{z} + W_{02}^{(2)}(s)\frac{\bar{z}^2}{2} + \cdots.$$

From (3.15) and (3.16), we have

$$g(z, \bar{z}) = \bar{q}^*(0) f_0(z, \bar{z}) = \bar{D}[f_1(0, x_t) + \bar{\gamma}^* f_2(0, x_t)]$$

$$= \overline{D}\left[-\frac{a_1}{3}x_{2t}^3(0) + \frac{a_1 b_1}{3}\int_{-\infty}^{0} F(-s)x_{2t}^3(0)ds\right]$$

$$+\overline{D}\bar{a}^* \left[-\frac{a_2}{3}x_{1t}^3(0) + \frac{a_2 b_2}{3}\int_{-\infty}^{0} F(-s)x_{1t}^3(0)ds\right] + h.o.t.,$$

and we obtain $g_{20} = g_{11} = g_{02} = 0$ and

$$g_{21} = 2\overline{D}\left[-a_1\overline{a}a^2 - \bar{a}^* a_2 + \frac{(a_1 b_1 a^2\overline{a} + a_2 b_2)\alpha^2}{\alpha + i\omega_0}\right].$$

Thus, we derive

$$c_1(0) = \frac{i}{2\omega_0}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}$$

$$= \frac{g_{21}}{2} = \overline{D}\left[-a_1\overline{a}a^2 - \bar{a}^* a_2 + \frac{(a_1 b_1 a^2\overline{a} + a_2 b_2)\alpha^2}{\alpha + i\omega_0}\right].$$

Then, we can easily compute the following parameters:

$$\mu_2 = -\frac{\mathrm{Re}\{c_1(0)\}}{\mathrm{Re}\{\lambda'(\alpha_0)\}},$$

$$\beta_2 = 2\,\mathrm{Re}(c_1(0)),$$

$$T_2 = -\frac{\mathrm{Im}\{c_1(0)\} + \mu_2\,\mathrm{Im}\{\lambda'(\alpha_0)\}}{\omega_0}.$$

These formulaes give a description of the Hopf bifurcation periodic solutions of (3.1) at $\alpha = \alpha_0$ on the center manifold. From the above discussion, we have the following result:

Theorem 3.1. *The periodic solution is supercritical* (*subcritical*), *if* $\mu_2 > 0\,(\mu_2 < 0)$; *the bifurcating periodic solutions are orbitally asymptotically stable with asymptotical phase* (*unstable*), *if* $\beta_2 < 0$ $(\beta_2 > 0)$; *the periodic of the bifurcating periodic solutions increase* (*decrease*), *if* $T_2 > 0\,(T_2 < 0)$.

4. Conclusion

In this paper, we have investigated local stability of the equilibrium $E_*(0, 0)$ and local Hopf bifurcation in a two-neuron network with distributed delays. We have showed that, if the conditions (H1), (H2), and (2.5)-(2.10) hold, the equilibrium $E_*(0, 0)$ of system (1.2) is asymptotically stable. We have also showed that, if the conditions (2.5)-(2.10) and (H1)-(H4) hold, as the parameter α passes through α_0, the equilibrium $E_*(0, 0)$ loses its stability and a Hopf bifurcations occurs at $E_*(0, 0)$, i.e., a periodic orbit bifurcates from the equilibrium $E_*(0, 0)$. At last, the direction of Hopf bifurcation and the stability of the bifurcating periodic orbits are discussed by applying the normal form theory and the center manifold theorem.

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